Lawvere-Tierney Sheafification in Homotopy Type Theory

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Introduction
The construction
Idea and context
Definitions
From types to separated types
From separated types to sheaves
Consequences
Future works
References

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Let's recall that in a topos, a Lawvere-Tierney topology is an idempotent map $\Omega \to \Omega$, preserving true and products. We notice that it corresponds to a left-exact modality on the subobject classifier Ω.

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1. Here, we call modality the same thing as in Type, but truncated to $n\text{-}\mathrm{Type}$
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Recall : Modalities

We use the same notion of modalities as in [Uni13, Section 7.7], but restricted to be on n-truncated types.

Definition

Let $n \geq -1$ be a truncation index. A left exact modality at level n is the data of

(i) *A predicate $P : \mathrm{Type}_n \to \mathrm{HProp}$*

(ii) *For every n-truncated type A, a n-truncated type $\bigcirc A$ such that $P(\bigcirc A)$*

(iii) *For every n-truncated type A, a map $\eta_A : A \to \bigcirc A$*

such that

(iv) *For every n-truncated types A and B, if $P(B)$ then*

$$\begin{cases} (\bigcirc A \to B) & \to & (A \to B) \\ f & \mapsto & f \circ \eta_A \end{cases}$$

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(v) for any $A : \text{Type}_n$ and $B : A \to \text{Type}_n$ such that $P(A)$ and $\prod_{x:A} P(Bx)$, then $P\left(\sum_{x:A} B(x)\right)$

(vi) for any $A : \text{Type}_n$ and $x, y : A$, if $\bigcirc A$ is contractible, then $\bigcirc(x = y)$ is contractible.

Conditions (i) to (iv) define a *reflective subuniverse*, (i) to (v) a *modality*.

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Recall: Sheafification in topos

Let j be a Lawvere-Tierney topology on a topos $\mathcal{T}$, with subobject classifier Ω.

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$$\begin{array}{ccc} T & \xrightarrow{\{\cdot\}_T} & \Omega^T \\ & & \big\downarrow{\scriptstyle j^T} \\ & & (\Omega_j)^T \end{array}$$

Send T to Ω^T via the singleton map, then postcompose with $j : \Omega \to \Omega_j$

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& \searrow \quad \nearrow & \\
& \mathfrak{a}(T) \stackrel{def}{=} \overline{\mathrm{Im}\left(j^T \circ \{\cdot\}_T\right)} &
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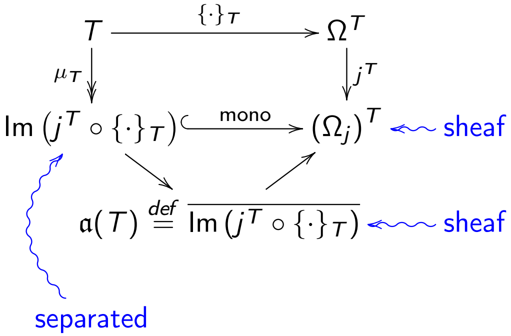
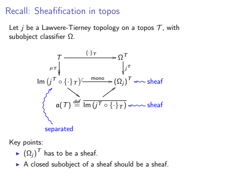


Key points:

- $(\Omega_j)^T$ has to be a sheaf.
- A closed subobject of a sheaf should be a sheaf.



The predicate "is n-modal" on homotopy level n will be "is a Lawvere-Tierney n-sheaf", and the required modality will be the n-sheafification.

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1. We do this by induction on the homotopy level n. At the moment, we don't know how to extend it to not truncated types
2. From *Sets in HoTT* (Rijke-Spitters), we know that n-Type can be seen as an object classifier. We will use this property ;
   HProp will be a common object classifier for all levels, and n-Type will be an object classifier for $-$-$\mathrm{Type}(n+1)$ sheafification.

Context

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Let $\bigcirc_{-1}$ be a left exact modality on HProp (homotopy level -1), $n \geqslant -1$ a truncation index, and $\bigcirc_n$ a left exact modality on n-Type (homotopy level n), coherent with $\bigcirc_{-1}$:

If T : HProp, then $\bigcirc_n T = \bigcirc_{-1} T$ where we still note T the image of T via the inclusion HProp $\hookrightarrow$ n-Type.

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There, by cumulativity, T can be seen as a n-Type.

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- Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobjects (embeddings) ? Solved
- The proof involves kernel pair of a surjection. How to generalize it ? In progress
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Dense subobject I

Definition

Let E be a type. The closure *of a subobject of E with m-truncated homotopy fibers (or m-subobject of E, for short), classified by $\chi : E \to m\text{-Type}$, is the m-subobject of E classified by $\bigcirc_m \circ \chi$.*
An m-subobject of E classified by χ is said to be closed *in E if it is equal to its closure, i.e, $\chi = \bigcirc_m \circ \chi$.*

Practically, a m-subobject of E is just $\{e : E \ \& \ \chi \ e\}$, and its closure is $\{e : E \ \& \ \bigcirc_m (\chi \ e)\}$.

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1. The closure operator is just postcomposition of characteristic with the modality.
2. A is closed in E if its closure is E.

Dense subobject II

Definition

Let E be a type, and $\chi : E \to m\text{-Type}$. The m-subobject of E classified by χ is dense *in E when its $\bigcirc_m$-closure is equivalent to χ_E, i.e,*

$$\forall e : E,\ \ \bigcirc_m (\chi\ e) \simeq \mathbf{1}.$$

Practically, a m-subobject A of E is dense if, from the $\bigcirc_m$ point of view, you cannot make a difference between A and E.

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Restriction

Definition

For any type E, characteristic map $\chi : E \to m\text{-Type}$ and $F : (n+1)\text{-Type}$, we define

$$\Phi_E^{\chi,m} : (E \to F) \to (\{e : E \ \& \ \chi\ e\} \to F)$$

as the map sending an arrow $f : E \to F$ to its restriction $f \circ \pi_1$.

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Requirements

We want a predicate on $(n+1)$-Type, which we call *sheaf property*, satisfying:

- if $\bigcirc_n$ is the identity modality, then everybody should be a sheaf
- a closed (-1)-subobject of a sheaf should be a sheaf
- the type of modal n-Type should be a sheaf
- if T is a sheaf, then $X \to T$ should be a sheaf, for any X

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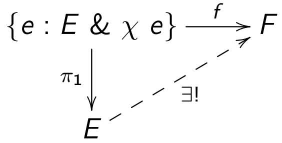
Sheaves

Following the topos-theoretic idea, we use:

Definition (Sheaves)

A type F of $(n+1)$-Type is a $(n+1)$-sheaf for any type E and all dense (-1)-subobject $\chi : E \to (-1)$-Type, $\Phi_E^{\chi,-1}$ is an equivalence. In other words, the dotted arrow exists and is unique.



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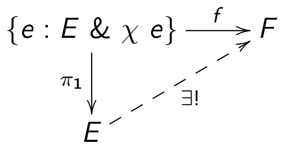
Sheaves

Following the topos-theoretic idea, we use:

Definition (Sheaves)

A type F of $(n+1)$-Type is a $(n+1)$-sheaf if it is separated, and for any type E and all dense (-1)-subobject $\chi : E \to (-1)\text{-Type}$, $\Phi_E^{\chi,-1}$ is an equivalence. In other words, the dotted arrow exists and is unique.



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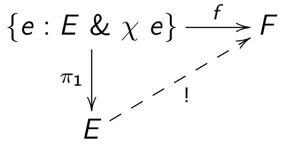
Separated type

Definition (Separated Type)

A type F in $(n+1)$-Type is separated *if for any type E, and all dense* n-subobject $\chi : E \to n\text{-Type}$, $\Phi_E^{\chi,n}$ *is an embedding. In other words, the dotted arrow, if exists, is unique.*



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Two steps

We will proceed in two steps:

(i) *separation:* From any T in $(n+1)\text{-Type}$, we construct its *free* separated object $\square_{n+1} T$.

(ii) *completion:* We add what is missing for the free separated type to be a sheaf by using closure.

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1. Not equivalent with + construction. We define the free separated object, while Grothendieck not.

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$$\begin{array}{ccc} T & \xrightarrow{\{\cdot\}_T} & n\text{-Type}^T \\ {\scriptstyle \mu_T}\downarrow & & \downarrow{\scriptstyle \bigcirc_n^T} \\ \square_{n+1}\, T & \longrightarrow & \left(n\text{-Type}^{\bigcirc}\right)^T \end{array}$$

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We note that, as μ_T is the surjection-embedding factorization, μ_T is indeed a surjection.

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$\square_{n+1}\ T$ can be given explicitly by

$$\begin{aligned} \square_{n+1}\ T &\overset{def}{=} \mathrm{Im}(\lambda\ t : T,\ \lambda\ t',\ \bigcirc_n (t = t')) \\ &\overset{def}{=} \textstyle\sum_{u : T \to n\text{-Type}^{\bigcirc}} \left\| \sum_{a : T} (\lambda t,\ \bigcirc_n (a = t)) = u \right\| . \end{aligned}$$

Let $T : (n+1)\text{-Type}$. We define $\square_{n+1}\ T$ as the image of $\bigcirc_n^T \circ \{\cdot\}_T$, as in
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We note that, as μ_T is the surjection-embedding factorization, μ_T is indeed a surjection.

At first, we prove that:

Proposition

For any $T : (n+1)\text{-Type}$, $\Box_{n+1}\, T$ *is separated.*

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1. That's indeed the least we can ask.
2. This actually is the hard part of the construction ; especially the universal property for the reflective subuniverse.

Sketch of proof

In topoi, the proof goes this way:

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- Then $T \times_{\Box_{n+1} T} T = \overline{\Delta}$, where $\Delta = \{(x, y) : T^2 \ \& \ x = y\}$. The following is a coequalizer

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Then, if Q is any separated type and $f : T \to Q$, it makes the diagram

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Introduction
The construction
Idea and context
Definitions
From types to separated types
From separated types to sheaves
Consequences
Future works
References

For any T in $(n+1)$-Type, $\bigcirc_{n+1} T$ is defined as the closure of $\square_{n+1}\ T$, seen as a subobject of $T \to n$-$\mathrm{Type}^{\bigcirc}$. $\bigcirc_{n+1} T$ can be given explicitly by

$$\bigcirc_{n+1} T \stackrel{def}{=} \sum_{u:T \to n\text{-}\mathrm{Type}^{\bigcirc}} \bigcirc_{-1} \left\| \sum_{a:T} (\lambda t,\ \ \bigcirc_n (a = t)) = u \right\|.$$

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As above, we first prove that:

Proposition

For any $T : (n+1)\text{-Type}$, $\bigcirc_{n+1} T$ is a sheaf.

It is true because of the requirement we asked about sheaves:

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Let $X : (n+1)\text{-Type}$ and U be a sheaf. If X embeds in U, and is closed in U, then X is a sheaf.

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2. This time, it's pretty easy...

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Let $T, Q : (n+1)\text{-}\mathrm{Type}$ such that Q is a sheaf. Let $f : T \to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\square_{n+1} f : \square_{n+1} T \to Q$.

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As all these steps are universal, the composition is.

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As all these steps are universal, the composition is.

again, the modality thing is just technical, and the left-exactness comes from the compatibility.

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Introduction
The construction
Idea and context
Definitions
From types to separated types
From separated types to sheaves
Consequences
Future works
References

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Introduction
The construction
Idea and context
Definitions
From types to separated types
From separated types to sheaves
Consequences
Future works
References

Universes

The construction can be written inductively:

$\bigcirc : \forall\ (n : nat),\ n\text{-Type} \to n\text{-Type}$

- $\bigcirc_{-1}$ is a left exact modality on HProp
- $\bigcirc_{n+1} \stackrel{def}{=} \lambda T : (n+1)\text{-Type},$

$$\sum_{u : T \to n\text{-Type}^{\bigcirc}} \bigcirc_{-1} \left\| \sum_{a : T} u = (\lambda t,\ \ \bigcirc_n (a = t)) \right\|$$

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Čech nerve

The main step to finish the construction is to define Čech nerve in HoTT, as well as the computation of their colimits.

We will rather try to define general simplicial objects.

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We will rather try to define general simplicial objects.

Simplicial types

Hugo Herbelin [Her14] gives an inductive definition of semi-simplicial types, which can probably be adapted to define simplicial types, but is quite unusable for n-types with $n \geqslant 4$.

Make a joke about the previous talk...

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Homotopy type system

One idea is to use homotopy type system, introduced by V.V., to see Type as a model category. Then, we should be able to formalize homotopy colimits in type theory.

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Introduction
The construction
Idea and context
Definitions
From types to separated types
From separated types to sheaves
Consequences
Future works
References

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