Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

29 June 2015

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

landari da anta a

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future wor

Reference

Lawvere-Tierney Sheafification in Homotopy Type Theory

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

29 June 2015

The construction

Idea and context

Definition

From separated types to sheaves
Consequences

Future works

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

-uture works

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
Introduction

Introduction

The construction
Idea and context
Definitions
From types to separated types
From separated types to sheaves
Consequences

References

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

5-06-29

Lawvere-Tierney Sheafification in Homotopy Type Theory -Introduction

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Introduction

Idea and context Separation Sheafification Consequences

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the *sheafification* process [MM92].

Lawvere-Tierney
Sheafification
in Homotopy Type
Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes
Nantes, France

5-06

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

ure works

Lawvere-Tierney Sheafification in Homotopy Type Theory

Introduction

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the sheafification process [MM92]. In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the *sheafification* process [MM92].

Then (Grothendieck) sheafification has been extended to higher topos theory [Lur09].

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

9

Introduction

Idea and context Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory -Introduction

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos. one can construct a new topos satisfying some new principles, using the sheafification process [MM92]. Then (Grothendisck) sheafification has been extended to higher topos theory [Lur09].

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the *sheafification* process [MM92].

Then (Grothendieck) sheafification has been extended to higher topos theory [Lur09].

We will present here a work-in-progress attempt to define an homotopy type theoretic version of this process. Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes

Nantes, France

9

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type Theory
Introduction

In set theory, one can change a model of ZFC into a new model of ZFC satisfying new principles, using the forcing construction [CD66].

Forcing has a topos-theoretic version: starting from a topos, one can construct a new topos satisfying some new principles, using the sheafification process [MM92]. Then (Grothendieck) sheafification has been extended to higher topos theory [Lur03].

We will present here a work-in-progress attempt to define an homotopy type theoretic version of this process.

The construction

Idea and context

Definitions

From types to separated types From separated types to sheaves Consequences

Future works

References

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

Introduction

Idea and context Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type 5-06-29 Theory -Introduction

The construction Idea and context Definitions From types to separated types From separated types to sheaves

Future works References

The construction

Idea and contex

Definition

From separated types to sheaves

Consequences

Future works

Referenc

Lawvere-Tierney Sheafification in Homotopy Type Theory

Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes
Nantes, France

ntroduction

The construction

Idea and context
Definitions
Separation
Sheafification
Consequences

ruture work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction

Introduction

The construction

Idea and content
Definitions

From types to separated types
From speated types to sheaves
Consequences

Future social

The construction

Idea and context

Definition:

From separated types to sheave: Consequences

Future works

Reference

Lawvere-Tierney Sheafification in Homotopy Type Theory

Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes
Nantes, France

ntroduction

he construction

Idea and context
Definitions
Separation
Sheafification
Consequences

Future works

Reference

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context

The construction

The construction

Max and context
Definitions
From typics to separated typics
From typics to subsected
Consequences
Consequences
Future such as

References

Let's recall that in a topos, a Lawvere-Tierney topology is an idempotent map $\Omega \to \Omega$, preserving true and products. We notice that it corresponds to a left-exact modality on the subobject classifier Ω .

Then, the sheafification process extend this modality to the whole topos.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

ntroduction

The construction

Separation
Sheafification
Consequences

ire works

o c

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context

Let's recall that in a topos, a Lawarer-Tierney topology in an idempotent map $\Omega \to \Omega_c$ preserving true and product. We notice that it corresponds to a left-exact modality on the subobject classifier Ω .

Then, the shealification process extend this modality to the whole topos.

1. Here, we call modality the same thing as in Type, but truncated to *n*-Type

2. Sets in HoTT (Rijke-Spitters) tells us we can view HProp as an object classifier : Ω will HProp, and the topos HSet

Let's recall that in a topos, a Lawvere-Tierney topology is an idempotent map $\Omega \to \Omega$, preserving true and products. We notice that it corresponds to a left-exact modality on the subobject classifier Ω .

Then, the sheafification process extend this modality to the whole topos.

We want to follow this idea: from a left exact modality on HProp, we will define a left exact modality on all (finite) homotopy levels, by induction on this level.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

The construction

Separation Sheafification Consequences

References

Lawvere-Tierney Sheafification in Homotopy Type Theory
The construction
Idea and context

Let's recall that in a topos, a Lawvere-Tierney topology is an idempotent map $\Omega \to \Omega$, preserving true and products. We notice that it corresponds to a left-exact modality on the subobject classifier Ω .

Then, the sheafification process extend this modality to the whole topos.

> We want to follow this idea : from a left exact modality on HProp, we will define a left exact modality on all (finite) homotopy levels, by induction on this level.

- 1. Here, we call modality the same thing as in Type, but truncated to *n*-Type
- 2. Sets in HoTT (Rijke-Spitters) tells us we can view HProp as an object classifier : Ω will HProp, and the topos HSet

Recall: Modalities

We use the same notion of modalities as in [Uni13, Section 7.7], but restricted to be on n-truncated types.

Definition

Let $n \ge -1$ be a truncation index. A left exact modality at level n is the data of

- (i) A predicate $P: \mathsf{Type}_n \to \mathsf{HProp}$
- (ii) For every n-truncated type A, a n-truncated type $\bigcirc A$ such that $P(\bigcirc A)$
- (iii) For every n-truncated type A, a map $\eta_A:A\to \bigcirc A$ such that
- (iv) For every n-truncated types A and B, if P(B) then

$$\left\{ \begin{array}{ccc} (\bigcirc A \to B) & \to & (A \to B) \\ f & \mapsto & f \circ \eta_A \end{array} \right.$$

is an equivalence.

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introduction

ne construction

Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context
Recall: Modalities

Recall : Modalities
We use the same notion of modalities as in [InVII.5 Section 7.7]. For matrices to be on extraorated types. Definition:
Let $n \ge -1$ be a transaction index. A left exact modality at least $n \ge -1$ be a transaction index. A left exact modality at least $n \ge -1$ be a transaction index. A left exact modality at least $n \ge -1$ be a transaction of $n \ge -1$. A predicate $n \ge -1$ $n \ge -1$ n

such that $P(\bigcirc A)$ (iii) For every n-truncated type A, a map $\eta_A : A \rightarrow \bigcirc A$ such that (iv) For every n-truncated types A and B, if P(B) then

(iv) For every n-truncated types A and B, if P(B) the $\int (\bigcirc A \rightarrow B) \rightarrow (A \rightarrow B)$

 $\begin{cases} f \mapsto f \circ \eta_A \end{cases}$ is an equivalence.

- (v) for any A : Type, and B : $A \rightarrow \text{Type}_n$ such that P(A)and $\prod_{x \in A} P(Bx)$, then $P(\sum_{x \in A} B(x))$
- (vi) for any A: Type, and x, y : A, if $\bigcirc A$ is contractible, then $\bigcirc(x=y)$ is contractible.

Conditions (i) to (iv) define a reflective subuniverse, (i) to (v) a modality.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

Idea and context

Separation Sheafification

Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory The construction

-Idea and context

(v) for any A: Type, and B: A → Type, such that P(A) and $\prod_{x \in A} P(Bx)$, then $P(\sum_{x \in A} B(x))$

Conditions (i) to (iv) define a reflective subuniverse. (i) to

Let j be a Lawvere-Tierney topology on a topos \mathcal{T} , with subobject classifier Ω .

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introduction

The construction

Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type

Theory

The construction

Idea and context

Recall: Sheafification in topos

Let j be a Lawvere-Tierney topology on a topos \mathcal{T} , with subobject classifier Ω .

$$T \xrightarrow{\{\cdot\}_T} \Omega^T \downarrow_{j^T} (\Omega_j)^T$$

Send T to Ω^T via the singleton map, then postcompose with $j:\Omega\to\Omega_j$

Lawvere-Tierney
Sheafification
in Homotopy Type
Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes
Nantes
Nantes
Nantes
Nantes

Introduction

e construction

Idea and context Definitions Separation Sheafification Consequences

Future work

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context
Recall: Sheafification in topos

Recall: Sheaffication in topos

Let J be a Lawver-Terrey topology on a topos T, with sub-object classifier Ω . $T \frac{1r}{(\Omega_J)^T}$ Q_J^T Sand T to Ω^T with the similar mean, then nontoneous

Let j be a Lawvere-Tierney topology on a topos \mathcal{T} , with subobject classifier Ω .

$$T \xrightarrow{\mu_{T} \downarrow} \Omega^{T} \xrightarrow{\downarrow j^{T}} \Omega^{T}$$

$$\downarrow j^{T}$$

$$\text{Im } (j^{T} \circ \{\cdot\}_{T}) \xrightarrow{\text{mono}} (\Omega_{j})^{T}$$

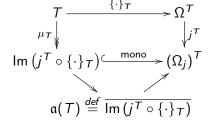
Compute the image of this map: it is a subobject of $(\Omega_i)^T$

Lawvere-Tierney Sheafification in Homotopy Type Lawvere-Tierney Sheafification Theory in Homotopy Type └─The construction Theory -Idea and context Kevin Quirin and Nicolas Tabareau Recall: Sheafification in topos Inria. Mines de Nantes Nantes, France

Recall: Sheafification in topos Let i be a Lawvere-Tierney topology on a topos T, with subobject classifier Ω. Compute the image of this map: it is a subobject of (Ω_i)

Idea and context Separation Sheafification Consequences

Let j be a Lawvere-Tierney topology on a topos \mathcal{T} , with subobject classifier Ω .



Close this subobject

Lawvere-Tierney
Sheafification
in Homotopy Type
Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes
Nantes, France

ntroduction

e construction

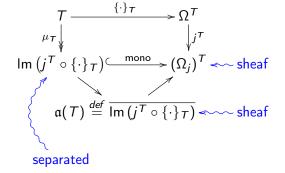
Idea and context Definitions Separation Sheafification Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context
Recall: Sheafification in topos

Let j be a Lawvere-Tierney topology on a topos \mathcal{T} , with subobject classifier Ω .



Key points:

- $\triangleright (\Omega_j)^T$ has to be a sheaf.
- A closed subobject of a sheaf should be a sheaf.

Lawvere-Tierney
Sheafification
in Homotopy Type
Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes

Lawvere-Tierney Sheafification in Homotopy Type
Theory

Theory

Light Construction

Recall: Sheafification in topos

Nantes, France

Idea and context
Definitions
Separation
Sheafification
Consequences



The predicate "is n-modal" on homotopy level n will be "is a Lawvere-Tierney n-sheaf", and the required modality will be the n-sheafification.

Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Lawvere-Tierney

Introduction

Idea and context
Definitions
Separation
Sheafification
Consequences

ire works

Theory
The construction
Idea and context

Lawvere-Tierney Sheafification in Homotopy Type

The predicate "is n-modal" on homotopy level n will be "is a Lawvere-Tierney n-sheaf", and the required modality will be the n-sheafification.

- 1. We do this by induction on the homotopy level *n*. At the moment, we don't know how to extend it to not truncated types
- From Sets in HoTT (Rijke-Spitters), we know that n-Type can be seen as an object classifier. We will use this property;
 HProp will be a common object classifier for all levels, and n-Type will be an object classifier for --Type(n+1) sheafification.

Context

We work in homotopy type theory, i.e, Martin-Löf type theory, with univalence axiom (thus functional extensionality) and higher inductive types (although at the moment, we only need propositional truncation).

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

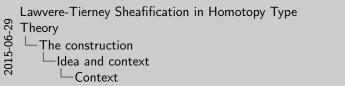
landari da anta an

he construction

Idea and context Definitions Separation Sheafification Consequences

Future work

Reference



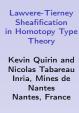
Context

We work in homotopy type theory, i.e, Martin-Löf type theory, with univalence axiom (thus functional extensionality) and higher inductive types (although at the moment, we only need propositional truncation).

Context

Let \bigcirc_{-1} be a left exact modality on HProp (homotopy level -1), $n\geqslant -1$ a truncation index, and \bigcirc_n a left exact modality on n-Type (homotopy level n), coherent with \bigcirc_{-1} :

If T: HProp, then $\bigcirc_n T = \bigcirc_{-1} T$ where we still note T the image of T via the inclusion HProp $\hookrightarrow n$ -Type.



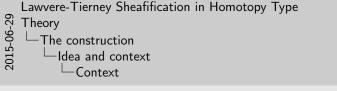
Inducation at a

The construction

Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Future work



Let \bigcirc_{-1} be a left exact modality on HProp (homotopy level -1), $n \geqslant -1$ a truncation index, and \bigcirc_n a left exact modality on e^{-1} Yevin (homotopy level n). coherent with \bigcirc_{-1} :

Context

modulity on n-Type (homotopy level n), coherent with \bigcirc_{-1} : If T: HProp, then $\bigcirc_n T = \bigcirc_{-1} T$ where we still note T the image of T via the inclusion HProp $\hookrightarrow n$ -Type.

There, by cumulativity, T can be seen as a n-Type.

When generalizing construction in topos, several questions arises:

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Idea and context Separation Sheafification Consequences

Questions Lawvere-Tierney Sheafification in Homotopy Type 2015-06-29 Theory When generalizing construction in topos, several questions The construction Idea and context -Questions

When generalizing construction in topos, several questions arises:

▶ Do we generalize subobjects as *n*-subobjects (maps with *n*-truncated fibers) or (-1)-subobjects (embeddings)?

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introductio

he construction

Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Reference

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context
Questions

Questions

When generalizing construction in topos, several questions arises:

Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobjects

When generalizing construction in topos, several questions arises:

- ▶ Do we generalize subobjects as *n*-subobjects (maps with *n*-truncated fibers) or (-1)-subobjects (embeddings)?
- ► The proof involves kernel pair of a surjection. How to generalize it ?

Lawvere-Tierney
Sheafification
in Homotopy Type
Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

lanca di carta i

he construction

Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

eferences

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context
Questions

Questions

When generalizing construction in topos, several questions arises:

- Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobjects (embeddines)?
- The proof involves kernel pair of a surjection. How to generalize it?

When generalizing construction in topos, several questions arises:

- ▶ Do we generalize subobjects as *n*-subobjects (maps with *n*-truncated fibers) or (-1)-subobjects (embeddings)?
- ▶ The proof involves kernel pair of a surjection. How to generalize it?
- ▶ Do we use usual image, or a *n*-image arising from *n*-connected/*n*-truncated factorization system ?

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes

Nantes, France

Idea and context

Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory The construction Idea and context -Questions

Questions

When generalizing construction in topos, several questions

▶ Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (-1)-subobiects

The proof involves kernel pair of a surjection. How to

n-connected/n-truncated factorization system

When generalizing construction in topos, several questions arises:

- ▶ Do we generalize subobjects as *n*-subobjects (maps with *n*-truncated fibers) or (-1)-subobjects (embeddings)? Solved
- ► The proof involves kernel pair of a surjection. How to generalize it ? In progress
- ▶ Do we use usual image, or a *n*-image arising from *n*-connected/*n*-truncated factorization system ? Solved

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes

Nantes, France

ne construction

Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Idea and context
Questions

Questions

When generalizing construction in topos, several questions arises:

 Do we generalize subobjects as n-subobjects (maps with n-truncated fibers) or (−1)-subobjects

➤ The proof involves kernel pair of a surjection. How to

 Do we use usual image, or a n-image arising from n-connected/n-truncated factorization system? Solved

The construction

Idea and contex

Definitions

From types to separated types From separated types to sheaves Consequences

Future works

Reference

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

ntroduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

-uture works

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions

The construction
Most and context
Definitions
From typic to separated types
From suparated types to sheaves
Consequences
Future works
References

Dense subobject I

Definition

Let E be a type. The closure of a subobject of E with m-truncated homotopy fibers (or m-subobject of E, for short), classified by $\chi: E \to m$ -Type, is the m-subobject of E classified by $\bigcirc_m \circ \chi$.

An m-subobject of E classified by χ is said to be closed in E if it is equal to its closure, i.e, $\chi = \bigcirc_m \circ \chi$.

Practically, a *m*-subobject of *E* is just $\{e : E \& \chi e\}$, and its closure is $\{e : E \& \bigcirc_m (\chi e)\}$.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de

> Nantes Nantes, France

Introduction

The constructio
Idea and context
Definitions
Separation
Sheafification
Consequences
Future works

eferences

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions
Definitions
Dense subobject

Dense subobject I

Definition

Let E be a type. The closure of a subobject of E with m-trancated homotopy flows (or m-subobject of E, for short), classified by $\chi: E \to m$ -Type, is the m-subobject of E classified by $\chi: E \to m$ -Type, is the m-subobject of E classified by χ is said to be closed in E if it is equal to its closure, i.e. $\chi: = C_{\infty} \circ \chi$. Practically, a m-subobject of E is just $\{e: E \land E \land \chi: e\}$, and its closure is $\{e: E \land C_{\infty} \land \chi: e\}$.

- 1. The closure operator is just postcomposition of characteristic with the modality.
- 2. A is closed in E if its closure is E.

Dense subobject II

Definition

Let E be a type, and $\chi: E \to m\text{-}\mathrm{Type}$. The m-subobject of E classified by χ is dense in E when its \bigcirc_m -closure is equivalent to χ_E , i.e,

$$\forall e : E, \bigcirc_m (\chi e) \simeq 1.$$

Practically, a m-subobject A of E is dense if, from the \bigcirc_m point of view, you cannot make a difference between A and E.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

to a second constraint

The construction
Idea and context
Definitions
Separation
Sheafification

Consequences
Future works

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions
Dense subobject

Dense subobject II

Definition Let E be a type, and $\chi : E \rightarrow m$ -Type. The m-subobject of E classified by χ is dense in E when its \bigcirc_{m} -closure is equivalent to χ_E , i.e, $\forall e : E, \bigcirc_m \{\chi, e\} \simeq 1$.

**. L, ON((*) - 1.

Practically, a m-subobject A of E is dense if, from the \bigcirc_m point of view, you cannot make a difference between A and

Restriction

Definition

For any type E, characteristic map $\chi: E \to m$ -Type and F: (n+1)-Type, we define

$$\Phi_E^{\chi,m}: (E \to F) \to (\{e : E \& \chi e\} \to F)$$

as the map sending an arrow $f: E \to F$ to its restriction $f \circ \pi_1$.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau

Inria, Mines de Nantes Nantes

landari da anta an

The construction
Idea and context
Definitions
Separation

Sheafification Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions
Restriction

Restriction

For any type E, characteristic map $\chi : E \to m$ -Type and F : (n+1)-Type, we define $\Phi_E^{F,0} : (E \to F) \to (\{e : E \& \chi e\} \to F)$ as the map sending an arrow $f : E \to F$ to its restriction

Requirements

We want a predicate on (n + 1)-Type, which we call *sheaf* property, satisfying:

- ightharpoonup if ightharpoonup is the identity modality, then everybody should be a sheaf
- \blacktriangleright a closed (-1)-subobject of a sheaf should be a sheaf
- ▶ the type of modal *n*-Type should be a sheaf
- if T is a sheaf, then $X \to T$ should be a sheaf, for any Χ

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes

Nantes, France

Idea and context Definitions Separation Sheafification

Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory 2-00 -The construction **Definitions** -Requirements

Requirements

We want a predicate on (n+1)-Type, which we call sheaf property, satisfying if ○_n is the identity modality, then everybody should

- ➤ a closed (-1)-subobject of a sheaf should be a sheal
- the type of modal n-Type should be a sheaf
- if T is a sheaf, then X → T should be a sheaf, for any

Requirements

We want a predicate on (n + 1)-Type, which we call *sheaf* property, satisfying:

- if \bigcirc_n is the identity modality, then everybody should be a sheaf
- ightharpoonup a closed (-1)-subobject of a sheaf should be a sheaf
- ▶ the type of modal *n*-Type should be a sheaf
- ▶ if $T: X \to (n+1)$ -Type such that any Tx is a sheaf, then $\prod_{x \in X} Tx$ should be a sheaf.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

and disease and

The constructio
Idea and context
Definitions
Separation
Sheafification
Consequences

-uture work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions
Requirements

Requirements

We want a predicate on (n + 1)-Type, which we call sheaf property, satisfying: ► if ○a is the identity modality, then everybody should

- if On is the identity modality, then everybody be a sheaf
- a closed (−1)-subobject of a sheaf should be a sheaf
- the type of modal n-Type should be a sheaf
 if T: X → (n+1)-Type such that any Tx is a sheaf.
- If I: X → (n+1)-Type such that any I x is a then ∏_{x:X} T x should be a sheaf.

Sheaves

Following the topos-theoretic idea, we use:

Definition (Sheaves)

A type F of (n+1)-Type is a (n+1)-sheaf for any type E and all dense (-1)-subobject $\chi: E \to (-1)$ -Type, $\Phi_E^{\chi,-1}$ is an equivalence. In other words, the dotted arrow exists and is unique.

$$\{e: E \& \chi e\} \xrightarrow{f} I$$

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes

Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Reference

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions
Sheaves

Sheaves

Following the topos-thoretic idea, we use:

Definition (Sheaves)

App $\varepsilon = f(n+1)$ -Type is a (n+1)-sheaf for any type E and all dense (-1)-sheaf points $v \in (-1)$ -Type, $v \in V$.

As a singularity, in other words, the detailed arrow seeks and it among $v \in V$. $v \in E$ at $v \in V$.

- 1. Here, we take (-1)-subobjects, because we want every type to be a sheaf for the identity modality.
- 2. The conditions are not satisfied that way; sheaves are not stable by dependent products.

Sheaves

Following the topos-theoretic idea, we use:

Definition (Sheaves)

A type F of (n+1)-Type is a (n+1)-sheaf if it is separated, and for any type E and all dense (-1)-subobject $\chi: E \to (-1)$ -Type, $\Phi_E^{\chi,-1}$ is an equivalence. In other words, the dotted arrow exists and is unique.

$$\{e: E \& \chi e\} \xrightarrow{f} F$$

$$\begin{bmatrix} \pi_1 \\ \downarrow \\ E \end{bmatrix}$$

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes

Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Reference

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions
Sheaves

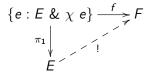
Sheaves
Following the topos-theoretic idea, we use:
Definition (Sheaves) $A_0 \in \mathcal{A}(n+1)$ -plotted if it is expected, and for any topo if and detune (-1)-included as expected, and for any topo if and of detune (-1)-included as excellent areas exists and is unique. It is obtained as the detail areas exists and is unique. If it is detailed areas exists and is unique.

- 1. Here, we take (-1)-subobjects, because we want every type to be a sheaf for the identity modality.
- 2. The conditions are not satisfied that way; sheaves are not stable by dependent products.

Separated type

Definition (Separated Type)

A type F in (n+1)-Type is separated if for any type E, and all dense n-subobject $\chi: E \to n$ -Type, $\Phi_E^{\chi,n}$ is an embedding. In other words, the dotted arrow, if exists, is unique.



Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introductio

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Definitions
Separated type

Separated type

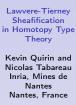
Definition (Separated Type) A type F in (n+1)-Type is separated if for any type E, and all dense n-subobject $\chi : E \to n$ -Type, $\Phi_E^{\Delta n}$ is an embedding. In other words, the dotted arrow, if exists, in

المستميم من الم

Two steps

We will proceed in two steps:

- (i) separation: From any T in (n+1)-Type, we construct its free separated object $\square_{n+1} T$.
- (ii) *completion:* We add what is missing for the free separated type to be a sheaf by using closure.



Inria, Mines di Nantes Nantes, Franco Introduction
The construction Idea and context Definitions Separation Sheafification Consequences
Future works
References

Lawvere-Tierney Sheafification in Homotopy Type

Theory

The construction

Definitions

Two steps

Two steps

1. Not equivalent with + construction. We define the free separated object, while Grothendieck not.

Introduction

The construction

Idea and conte

From types to separated types

From separated types to sheaves Consequences

Future works

Reference

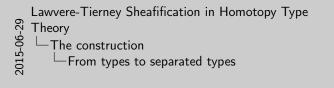
Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

ntroduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

uture works

Reference



Introduction

The construction

Mos and context
Defending

From types to especiated types

From types to provide types

Consequence

Future works

Let T: (n+1)-Type. We define $\square_{n+1} T$ as the image of $\bigcap_{n=1}^{T} \circ \{\cdot\}_{T_n}$, as in

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France Idea and context Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From types to separated types



We note that, as μ_T is the surjection-embedding factorization, μ_T is indeed a surjection.

Let T: (n+1)-Type. We define $\square_{n+1} T$ as the image of $\bigcap_{n=1}^{T} \circ \{\cdot\}_{T_n}$, as in

$$T \xrightarrow{\{r\}T} n\text{-Type}^T$$

$$\downarrow^{\bigcap_n^T}$$

$$\square_{n+1} T \longrightarrow (n\text{-Type}^{\bigcirc})^T$$

where $\{\cdot\}_T$ is the singleton map $\lambda(t:T)$, $\lambda(t':T)$, t=t'. \square_{n+1} T can be given explicitly by

$$\Box_{n+1} T \stackrel{\text{def}}{=} \operatorname{Im}(\lambda \ t : T, \ \lambda \ t', \ \bigcirc_n (t = t'))$$

$$\stackrel{\text{def}}{=} \sum_{u : T \to n\text{-Type}} \Vert \sum_{a : T} (\lambda t, \ \bigcirc_n (a = t)) = u \Vert.$$

Lawvere-Tierney
Sheafification
in Homotopy Type
Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes
Nantes, France

The construction

Separation Sheafification

Future works

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From types to separated types

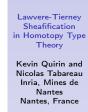


We note that, as μ_T is the surjection-embedding factorization, μ_T is indeed a surjection.

At first, we prove that:

Proposition

For any T:(n+1)-Type, $\square_{n+1} T$ is separated.



Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From types to separated types

At first, we prove that:

For any T:(n+1)-Type, $\square_{n+1} T$ is separated.

Proposition

1. That's indeed the least we can ask.

At first, we prove that:

Proposition

For any T: (n+1)-Type, $\square_{n+1} T$ is separated.

Then, we want

Theorem

 (\square_{n+1}, μ) defines a modality on (n+1)-Type.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

landari da aktorio

The construction
Idea and context
Definitions
Separation
Sheafification

Consequences
Future works

ruture work

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From types to separated types

At first, we prove that: Proposition For sig T : (n+1)-Type, $\square_{\alpha+1} T$ is separated. Then, we want Theorem $(\square_{\alpha+1}, p)$ defines a modality or (n+1)-Type.

- 1. That's indeed the least we can ask.
- 2. This actually is the hard part of the construction; especially the universal property for the reflective subuniverse.

In topoi, the proof goes this way:

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introduction

Idea and context Definitions Separation Separation Consequences Future works References Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From types to separated types
Sketch of proof

In topoi, the proof goes this way:

 \blacktriangleright μ_T is a surjection, thus it coequalizes its kernel pair

$$T \times_{\square_{n+1} T} T \xrightarrow{\pi_1} T \xrightarrow{\mu_T} \square_{n+1} T$$

Lawvere-Tierney
Sheafification
in Homotopy Type
Theory

Kevin Quirin and
Nicolas Tabareau
Inria, Mines de
Nantes

Lawvere-Tierney
Theory

The construct
From type
Sketch

Nantes Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation

Sheafification Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From types to separated types
Sketch of proof

Sketch of proof in topol, the proof goes this way: $+ \mu_T \text{ is a surjection, thus it conqualizes its kernel pair } T \times_{Const.T} T \xrightarrow{Const.T} T \xrightarrow{Const.T} T \xrightarrow{Const.T} T$

In topoi, the proof goes this way:

 \blacktriangleright μ_T is a surjection, thus it coequalizes its kernel pair

$$T \times_{\square_{n+1} T} T \xrightarrow{\pi_1} T \xrightarrow{\mu_T} \square_{n+1} T$$

► Then $T \times_{\square_{n+1} T} T = \overline{\Delta}$, where $\Delta = \{(x,y) : T^2 \& x = y\}$. The following is a coequalizer

$$\overline{\Delta} \xrightarrow{\pi_1} T \xrightarrow{\mu_T} \Box_{n+1} T$$

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

to an alternative

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Tuture wor

Lawvere-Tierney Sheafification in Homotopy Type
Theory
Theory
The construction
From types to separated types
Sketch of proof

Sleetch of proof in topo; the proof give this way: * p is a superior, then it compulsate its kernel pair $T = c_{i+1} T = \frac{1}{2} T = \frac{1}{2} T = c_{i+1} T$ * Then $T = c_{i+1} T = T = \frac{1}{2} T = c_{i+1} T = c_{i+1} T$ $\Delta = \{c_{i+1}\} T \neq \Delta = T = T = c_{i+1} T = T = c_{i+1} T = C_{i+1} T = T = C_{i+1} T = C_{i+1} T = T = C_{i+1} T = C_{i+1}$

 $\overline{\Delta} \xrightarrow{\pi_3} T \xrightarrow{\mu_T} \Box_{\sigma+1} T$

Then, if Q is any separated type and $f: T \to Q$, it makes the diagram

$$\overline{\Delta} \xrightarrow{\pi_1} T \xrightarrow{f} Q$$

commute, thus f factors through $\square_{n+1} T$.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

Idea and context Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory —The construction From types to separated types -Sketch of proof

Then, if Q is any separated type and $f: T \rightarrow Q$, it makes $\overline{\Delta} \xrightarrow{\pi_1} T \xrightarrow{f} O$ commute, thus f factors through \square_{n+1} T

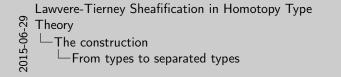
Sketch of proof

We would like to use the same idea, replacing the kernel pair by the Čech nerve.

At the moment, we only assumed as an axiom that surjections are colimits of their Čech nerves, seen as graphs. It allows us to finish the proof.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

Idea and context Separation Sheafification Consequences



We would like to use the same idea, replacing the kernel pair by the Čech nerve.

surjections are colimits of their Čech nerves, seen as graphs.

Introduction

The construction

Idea and contex

From types to separated type

From separated types to sheaves

Consequences

Future works

Reference

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Reference

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From separated types to sheaves

The construction

The construction

Most and context

Even types to essential types

From separated types to sheaves

Consequences

Future works

References

For any T in (n+1)-Type, $\bigcirc_{n+1} T$ is defined as the closure of $\square_{n+1} T$, seen as a subobject of $T \to n$ -Type^{\bigcirc}. $\bigcirc_{n+1} T$ can be given explicitly by

$$\bigcirc_{n+1} T \stackrel{\text{def}}{=} \sum_{u: T \to n\text{-Type}^{\bigcirc}} \bigcirc_{-1} \left\| \sum_{a: T} (\lambda t, \bigcirc_n (a=t)) = u \right\|.$$

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

Idea and context Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory The construction From separated types to sheaves

For any T in (n + 1)-Type, $\bigcirc_{n+1} T$ is defined as the closure of $\square_{n+1} T$, seen as a subobject of $T \rightarrow n\text{-}\text{Type}^-$ On T can be given explicitly by

$$\bigcirc_{n+1} T \stackrel{def}{=} \sum_{u: T \to n \text{-Type}^{\circ}} \bigcirc_{-1} \left\| \sum_{a: T} (\lambda t, \bigcirc_n (a = t)) = u \right\|.$$

As above, we first prove that:

Proposition

For any T: (n+1)-Type, $\bigcirc_{n+1} T$ is a sheaf.

It is true because of the requirement we asked about sheaves:

Lemma

Let X : (n+1)-Type and U be a sheaf. If X embeds in U, and is closed in U, then X is a sheaf.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes

Nantes, France

Idea and context Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory The construction -From separated types to sheaves

Proposition For any T:(n+1)-Type, $\bigcirc_{n+1}T$ is a sheaf. It is true because of the requirement we asked about Let X: (n+1)-Type and U be a sheaf. If X embeds in U

1. Again, we need this.

As above, we first prove that:

Proposition

For any T: (n+1)-Type, $\bigcirc_{n+1} T$ is a sheaf.

It is true because of the requirement we asked about sheaves:

Lemma

Let X : (n+1)-Type and U be a sheaf. If X embeds in U, and is closed in U, then X is a sheaf.

Then:

Theorem

 (\bigcirc_{n+1}, ν) defines a left-exact modality.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de

Inria, Mines de Nantes Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From separated types to sheaves

As above, we first prove that: Proposition For any T: (n+1)-Type, $\bigcirc_{n+1}T$ is a sheaf.

Let X: (n+1)-Type and U be a sheaf. If X embeds in U

For any T: (n+1)-Type, On+1T is a sheaf. It is true because of the requirement we asked about sheaves:

> Then: Theorem

(O_{n+1}, ν) defines a left-exact modality.

1. Again, we need this.

2. This time, it's pretty easy...

Let T, Q: (n+1)-Type such that Q is a sheaf. Let $f: T \to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\square_{n+1} f: \square_{n+1} T \to Q$.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introductio

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Lawvere-Tierney Sheafification in Homotopy Type

Theory

The construction

From separated types to sheaves

Sketch of proof

Let T,Q:(n+1)-Type such that Q is a sheaf. Let $f:T\to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\square_{n+1}f:\square_{n+1}T\to Q$.

Sketch of proof

Let T, Q: (n+1)-Type such that Q is a sheaf. Let $f: T \to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\square_{n+1} f : \square_{n+1} T \to Q$.

But as $\bigcirc_{n+1} T$ is the closure of $\square_{n+1} T$, $\square_{n+1} T$ is dense into $\bigcirc_{n+1} T$, so the sheaf property of Q allows to extend

 $\square_{n+1} f$ to $\bigcirc_{n+1} f: \bigcirc_{n+1} T \to Q$.

As all these steps are universal, the composition is.

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

Idea and context Separation Sheafification Consequences

Lawvere-Tierney Sheafification in Homotopy Type Theory The construction From separated types to sheaves -Sketch of proof

Sketch of proof

Let T, Q: (n+1)-Type such that Q is a sheaf. Let separated; thus we can extend f to $\square_{n+1} f : \square_{n+1} T \rightarrow Q$ But as $\bigcirc_{n+1}T$ is the closure of $\square_{n+1}T$, $\square_{n+1}T$ is dense into O.... T. so the sheaf property of Q allows to extend $\square_{n+1} f$ to $\bigcirc_{n+1} f : \bigcirc_{n+1} T \rightarrow Q$. As all these steps are universal, the composition is.

Let T, Q: (n+1)-Type such that Q is a sheaf. Let $f: T \to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\square_{n+1} f: \square_{n+1} T \to Q$.

But as $\bigcirc_{n+1} T$ is the closure of $\square_{n+1} T$, $\square_{n+1} T$ is dense into $\bigcirc_{n+1} T$, so the sheaf property of Q allows to extend $\square_{n+1} f$ to $\bigcirc_{n+1} f: \bigcirc_{n+1} T \to Q$.

As all these steps are universal, the composition is.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de

> Nantes Nantes, France

Introductio

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

i didie worr

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
From separated types to sheaves
Sketch of proof

Let T,Q:(n+1)-Type such that Q is a sheaf. Let $f:T\to Q$. Because Q is a sheaf, it is in particular separated; thus we can extend f to $\Box_{n+1} f:\Box_{n+1} T\to Q$ but as $\Box_{n+1} f$. Time f is the closure of $\Box_{n+1} f$, $\Box_{n+1} f$. It is that $\Box_{n+1} f$, to the sheaf property of Q allows to extend $\Box_{n+1} f:\Box_{n+1} f:\Box_{n+1} f\to Q$. As all thisse steps are universal, the composition is.

Sketch of proof

again, the modality thing is just technical, and the left-exactness comes from the compatibility.

Introduction

The construction

Idea and context
Definitions
From types to separated types
From separated types to sheave
Consequences

Future works

Reference

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

ntroduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

-uture works

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Consequences

Introduction

The construction

Mas and context

Definitions

From sequented types

From sequented types to showes

Consequences

Futner works

Starting from the left-exact modality $\bigcirc_{-1}P=\neg\neg P$, this allows us to build a model satisfying excluded middle for HProp, without axiom.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Consequences

Starting from the left-exact modality $\bigcirc_{-1}P = \neg\neg P$, this allows us to build a model satisfying excluded middle for HProp, without axiom.

Starting from the left-exact modality $\bigcirc_{-1}P = \neg \neg P$, this allows us to build a model satisfying excluded middle for HProp, without axiom.

With the same modality $\neg\neg$, we hope to be able to formalize the proof of independence of continuum hypothesis (actually, just the consistence of $\neg HC$).

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

2015-06-

and the second

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future work

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
The construction
Consequences

Starting from the left-exact modality $\bigcirc_{-1}P = \neg\neg P$, this allows us to build a model satisfying excluded middle for HProp, without axiom.

With the same modality ---, we hope to be able to formalize the proof of independance of continuum hypothesis (actually, just the consistance of -HC).

Introduction

The construction

Idea and contex

Definition

From separated types to sheave:
Consequences

Future works

Reference

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

ntroduction

The construction
Idea and context
Definitions
Separation
Sheafification

Consequences
Future works

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
Future works

Introduction

The construction
fide and context
Definitions
From typics to separated types
From separated types to sheaves
Consequences

Future works

Universes

The construction can be written inductively:

$$\bigcirc$$
: \forall (n : nat), n -Type \rightarrow n -Type

- $\bullet \ \bigcirc_{-1}$ is a left exact modality on HProp
- $\bigcirc_{n+1} \stackrel{\text{def}}{=} \lambda T : (n+1)$ -Type,

$$\sum_{u:T\to n\text{-Type}^{\bigcirc}}\bigcirc_{-1}\left\|\sum_{a:T}u=(\lambda t,\bigcirc_{n}(a=t))\right\|$$

Here , the universe level increases strictly at each step, hence it is impossible to take the fixpoint: we would need universes to be indexed by (non-finite) ordinals.

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future works

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
—Future works

—Universes

Universes

The construction can be written inductively

 $\bigcirc: \forall (n : nat), n\text{-Type} \rightarrow n\text{-Type}$ $\bullet \bigcirc_{-1}$ is a left exact modality on HProp $\bullet \bigcirc_{n+1} \stackrel{\text{def}}{=} \lambda T : (n+1)\text{-Type},$

 $\sum_{u:T\to a\text{-Type}^{\circ}} \langle T:(s+1)\text{-Type}, \atop \sum_{u:T\to a\text{-Type}^{\circ}} \circlearrowleft_{-1} \left\| \sum_{x:T} u = (\lambda t, \circlearrowleft_{a} (x =$

Here , the universe level increases strictly at each step, hence it is impossible to take the fixpoint: we would nuniverses to be indexed by (non-finite) ordinals.

Čech nerve

The main step to finish the construction is to define Čech nerve in HoTT, as well as the computation of their colimits.

We will rather try to define general simplicial objects.

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

to an alternative

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future works

Lawvere-Tierney Sheafification in Homotopy Type
Theory
Future works

Cech nerve

Čech nerve

The main step to finish the construction is to define Čech nerve in HoTT, as well as the computation of their colimits. We will rather try to define general simplicial objects.

Simplicial types

Hugo Herbelin [Her14] gives an inductive definition of semi-simplicial types, which can probably be adapted to define simplicial types, but is quite unusable for n-types with $n \ge 4$.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

5-06

Introduction

The construction
Idea and context
Definitions
Separation
Sheafification
Consequences

Future works

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
—Future works

—Simplicial types

Make a joke about the previous talk...

Simplicial types

Hugo Herbelin [Her14] gives an inductive definition of semi-simplicial types, which can probably be adapted to define simplicial types, but is quite unusable for n-types with $n\geqslant 4$.

Homotopy type system

One idea is to use homotopy type system, introduced by V.V., to see Type as a model category. Then, we should be able to formalize homotopy colimits in type theory.

Lawvere-Tierney Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria. Mines de Nantes Nantes, France

2015-06-

Idea and context Separation Sheafification Consequences

Future works

Homotopy type system Lawvere-Tierney Sheafification in Homotopy Type Theory Future works able to formalize homotopy colimits in type theory -Homotopy type system

One idea is to use homotopy type system, introduced by V.V., to see Type as a model category. Then, we should be

Introduction

The construction

Idea and contex

Definition

From separated types to sheaves

Consequences

Future works

References

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

ntroduction

The construction Idea and context Definitions Separation

Sheafification Consequences Future works

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
References

Introduction

The construction

Mai and content

Disformation organized types

From separated types to sheaves

Consequence

Future work

Reference

- P.J. Cohen and M. Davis, *Set theory and the continuum hypothesis*, WA Benjamin New York, 1966.
- Hugo Herbelin, A dependently-typed construction of semi-simplicial types, Mathematical Structures in Computer Science (2014), to appear.
- Jacob Lurie, *Higher topos theory*, Annals of mathematics studies, Princeton University Press, Princeton, N.J., Oxford, 2009.
- Saunders MacLane and leke Moerdijk, *Sheaves in geometry and logic*, Springer-Verlag, 1992.
- Univalent Foundations Project, Homotopy type theory: Univalent foundations for mathematics, http://homotopytypetheory.org/book, 2013.

Sheafification in Homotopy Type Theory Kevin Quirin and Nicolas Tabareau Inria, Mines de Nantes Nantes, France

90

Lawvere-Tierney

ntroduction

The construction Idea and context Definitions Separation Sheafification Consequences

References

Lawvere-Tierney Sheafification in Homotopy Type
Theory
References

